FOURIER-MUKAI TRANSFORMATION ON ALGEBRAIC COBORDISM

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Abstract. We define a notion of Fourier-Mukai transform for abelian varieties on any oriented cohomology theory with $\mathbb{Q}$-coefficients $A^*_{\mathbb{Q}}$. We use it to produce a Beauville decomposition of $A^*_{\mathbb{Q}}$ and study its consequences, including a decomposition of the A-motive of an abelian variety.

1. Introduction

The Fourier transformation is a well-known operator in analysis that gives an isometry between the L^2-spaces of a real vector space and its dual. In [12], Mukai introduced an analogous notion for sheaves of modules over abelian varieties. Let X and $\hat{X}$ be an abelian variety and its dual. Using the normalized Poincaré bundle $\mathfrak{P}$ on $X \times \hat{X}$ Mukai defined a functor between the derived categories of the sheaves of modules over X and $\hat{X}$, and proved that this is an equivalence of categories. In [2], Beauville used his results to define such a functor on cohomology, K-theory and the Chow ring of X with similar properties in each theory. He studied these Fourier transformations in detail and proved interesting consequences, among which is a decomposition of the Chow ring of an abelian variety into the eigenspaces of the pullbacks associated to the morphisms of multiplication by n, for all integers n (see [3]). Deninger and Murre used Beauville and Mukai's work in [5] to give a decomposition of the diagonal in $CH(X \times X) \otimes \mathbb{Q}$, which induces a canonical decomposition of Chow motives of an abelian variety.

Levine and Morel introduced the notion of oriented cohomology theories on $\mathbf{Sm}_k$ (see [10] for the definition) and constructed the algebraic cobordism as the universal such theory. The Chow ring being an oriented cohomology theory, a natural question to ask is whether a functor with the usual properties of a Fourier-Mukai transformation can be defined in this more general setting. In this paper, we define a Fourier-Mukai transformation on any oriented cohomology theory with $\mathbb{Q}$-coefficients and study its consequences.

The key idea that helped us extend the definition of the Fourier-Mukai transformation is that, working with $\mathbb{Q}$-coefficients, any oriented cohomology theory A^* can be twisted to form another theory A^*_{ad} which has an additive formal group law. Also, when applied on an abelian variety X, the triviality of the tangent bundle on X implies the existence of an isomorphism of rings $A^*(X) \to A^*_{ad}(X)$. Using these ideas, we generalize the Chern character and produce a ring homomorphism $\mathcal{C} : K^0(X) \to A^*_{\mathbb{Q}}(X)$. Let $\mathcal{P}$ denote the class of $\mathfrak{P}$ in $K^0(X \times \hat{X})$. After recalling some background material in sections 2 and 3, we use the "A-correspondence" $\mathcal{C}(\mathcal{P})$ in section 4 to define the Fourier-Mukai transformation $\mathcal{F}^A : A^*_{\mathbb{Q}}(X) \to A^*_{\mathbb{Q}}(\hat{X})$, so that it generalizes the known Fourier-Mukai transformations. In section 5, we obtain analogues of all the properties of $\mathcal{F}$ listed in [2, Proposition 3]. As a consequence, in section 6, we get a Beauville decomposition of $A^*_{\mathbb{Q}}(X)$ for any oriented cohomology theory A^* and any abelian variety X, generalizing [3, Théorème].

Theorem 1.1. *Let X be an abelian variety of dimension g over k. Then, we have*

$$A^p_{\mathbb{Q}}(X) = \bigoplus_{s=2p-2g}^{2p} {A_{\mathbb{Q}}}^p_s(X),$$

where ${A_{\mathbb{Q}}}^p_s(X) := \left\{ x \in A^p_{\mathbb{Q}}(X) \,|\, \forall\, n \in \mathbb{Z},\ \mathbf{n}^* x = n^{2p-s} x \right\}$.

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We also improve the limits of the decomposition in case of algebraic cobordism. In section 7, we prove that there is a canonical decomposition of the A-motive (defined in [14, § 6]) of an abelian variety.

Theorem 1.2. *There is a canonical decomposition of the A-motive with $\mathbb{Q}$-coefficients of an abelian variety X of dimension g over k:*

$$h_A(X) = \bigoplus_{i=0}^{2g} h^i_A(X)\,,$$

where $h_A(X) = (X, \mathrm{id}_X, 0)$ is the motive of X, $h^i_A(X) = (X, p^A_i, 0)$ and the p^A_i's are such that $c_A(\mathbf{n}) \circ p^A_i = n^i p^A_i = p^A_i \circ c_A(\mathbf{n})$.

In view of [8, Theorem B], we have a Künneth isomorphism in $\mathrm{MU}_{\mathbb{Q}}$. We check that the decomposition of the cobordism motive of an abelian variety is a Künneth decomposition with respect to the canonical morphism from algebraic to complex cobordism. In section 8, we define an integral Fourier-Mukai transformation $\mathcal{F}^A : A^*(X) \to A^*(\hat{X})$ following the ideas in [2, Proposition 3′]. As an application of the properties of the Fourier-Mukai transformation, in section 9 we generalize a result of Bloch [4] to the case of cobordism cycles. Let $\mathcal{N}^*(A)$ be the group of numerically trivial cobordism cycles on A as defined in [1, Definition 3.1] and let $\star$ be the Pontrygin product on $\Omega^*(A)$. Then, we show

Proposition 1.3. $\mathcal{N}^{*\star(g+1)}_{\mathbb{Q}}(X) = (0)$.

As a corollary, we show that $\mathcal{A}^{*\star(g+1)}(X) = 0$ where $\mathcal{A}^*(X)$ denotes the subgroup of $\Omega^*(X)$ consisting of cobordism cycles algebraically equivalent to zero (see [7, § 3]).

2. Oriented cohomology theories and Algebraic cobordism

In [10], inspired by the work of Quillen on complex differentiable manifolds, Levine and Morel introduced the notion of an oriented cohomology theory: a contravariant functor A^* from $\mathbf{Sm}_k$ to graded rings together with a collection of push-forward maps f_* associated to projective morphisms. This family is meant to respect functoriality and to be compatible on cartesian squares with the pull-back maps g^* every time two morphisms f and g are transverse. Finally, the functor is supposed to satisfy both the projective bundle formula, which expresses the evaluation of A^* on a projective bundle in terms of that of the base, and the extended homotopy property, which requires $p^* : A^*(X) \to A^*(V)$ to be an isomorphism for every vector bundle $E \to V$ and every E-torsor $p : V \to X$. As one might expect, a morphism of oriented cohomology theories is a natural transformation of functors which is also compatible with the push-forward morphisms f_*. For the precise definition we refer the reader to [10, Definition 1.1.2]. Important examples of functors which are also oriented cohomology theories include the Chow ring CH^* and $K^0[\beta, \beta^{-1}] := K^0 \otimes \mathbb{Z}[\beta, \beta^{-1}]$, a graded version of the Grothendieck ring of vector bundles.

A relevant feature of oriented cohomology theories is that they allow a theory of Chern classes. Even though in order to establish it for any bundle $E \to X$ it is necessary to rely on the projective bundle formula, the first Chern class of a line bundle $L \to X$ can be defined as $c_1(L) := s^* s_* 1_X \in A^*(X)$, where s denotes the zero section of L. Once the first Chern class is available, one may consider how it relates to the tensor product. While for the Chow group one has $c_1(L \otimes M) = c_1(L) + c_1(M)$, the equality is not true in general for oriented cohomology theories and one is forced to replace the usual addition with a formal group law:

$$c_1(L \otimes M) = F_A(c_1(L), c_1(M))$$

for a certain $F_A \in A^*(k)[[u, v]]$. A commutative formal group law of rank one (R, F_R) consists of a ring R and a power series $F_R \in R[[u, v]]$ satisfying conditions which are analogues of those for the operation in a group. For instance, the analogue of the associative property reads

$$F_R(F_R(u, v), w) = F_R(u, F_R(v, w)) \in R[[u, v, w]]\ .$$

In [9], Lazard identified the universal such object $(\mathbb{L}, F)$ and proved that the ring of coefficients, now known as the Lazard ring, is isomorphic to $\mathbb{Z}[a_1, a_2, \ldots]$. In this context, the universality means that for every formal group law (R, F_R) there exists a unique ring homomorphism $\phi_{(R,F_R)} : \mathbb{L} \to R$ such that $\phi_{(R,F_R)}(F) = F_R$, where $\phi_{(R,F_R)}(F)$ stands for the power series obtained by applying $\phi_{(R,F_R)}$ to the individual coefficients of F. Since it will be needed later on, let us add that the Lazard ring can be made into a graded ring $\mathbb{L}^*$ by setting $\deg a_i = -i$.

Taking into consideration formal group laws makes it evident that the analogy with the situation in topology does not end with the introduction of oriented cohomology theories. In fact, in [15], Quillen proved that complex cobordism MU^* is universal among complex oriented cohomology theories, that $MU^*(pt) \simeq \mathbb{L}^*$ and finally that its formal group law is the universal one. From this perspective, the theory of algebraic cobordism Ω^*, developed in [10] by Levine and Morel, represents the exact analogue of MU^* as testified by the following two theorems.

Theorem 2.1 ([10, Theorem 1.2.6]). *Let k be a field of characteristic 0. Then, given any oriented cohomology theory A^* on $\mathbf{Sm}_k$, there is a unique morphism*

$$\nu_A : \Omega^* \to A^*$$

of oriented cohomology theories.

Theorem 2.2 ([10, Theorem 1.2.7]). *For any field k of characteristic 0, the canonical homomorphism classifying F_Ω*

$$\phi_\Omega : \mathbb{L}^* \to \Omega^*(k)$$

is an isomorphism.

Notice that, provided $\Omega^*(k)$ is identified with the Lazard ring via ϕ_Ω, the evaluation of ν_A on $\operatorname{Spec} k$ coincides with $\phi_{(A,F_A)}$ and as a consequence one has $\nu_A(F_\Omega) = F_A$.

Before we briefly illustrate the construction of algebraic cobordism, let us recall the definition of ordinary and multiplicative theories, together with the associated analogues of Theorem 2.1. An oriented cohomology theory A^* is said to be *ordinary* if $F_A = F_a$ is the additive formal group law, *i.e.* $F_A(u, v) = u + v$. If instead there exists $b \in A^*(k)$, such that $F_A(u, v) = u + v - b \cdot uv$, then both A^* and $F_A = F_m$ are said to be *multiplicative*. In case b happens to be a unit we will also say that A^* is *periodic*. The following results describe the universal theories of these types, relating them to Ω^*.

Theorem 2.3 ([10, Theorem 1.2.2 & 7.1.4 (2)]). *Let k be a field of characteristic 0. The theory $\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}$ obtained by tensoring along $\phi_{(\mathbb{Z},F_a)}$ is isomorphic to CH^*. Moreover, if A^* is an ordinary theory there exists a unique morphism of oriented cohomology theories*

$$\nu_A^{CH} : CH^* \to A^*.$$

Theorem 2.4 ([10, Theorem 1.2.3 & 7.1.4 (1)]). *Let k be a field admitting resolution of singularities. The theory $\Omega^* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}]$ obtained by tensoring along $\phi_{(\mathbb{Z}[\beta,\beta^{-1}],F_m)}$ is isomorphic to $K^0[\beta, \beta^{-1}]$. Furthermore, if A^* is a multiplicative periodic theory there exists a unique morphism of oriented cohomology theories*

$$\nu_A^{K^0[\beta,\beta^{-1}]} : K^0[\beta, \beta^{-1}] \to A^*.$$

Now we briefly sketch the construction of algebraic cobordism, following the presentation of [10, Chapter 2]. As a group $\Omega^*(X)$ is obtained from the free group generated by the isomorphism classes of cobordism cycles

$$[f : Y \to X, L_1, \dots, L_r]$$

where f is a projective morphism with $Y \in \mathbf{Sm}_k$ and $\{L_1, \dots, L_r\}$ is a (possibly empty) family of line bundles over Y. Such a cycle has dimension $d = \dim_k Y - r$ and codimension $\dim_k X - d$. On this group one successively imposes three families of relations, each arising from a different geometric condition. For what concerns the multiplicative structure, it is achieved by constructing pull-backs for l.c.i. morphisms through an adaptation of the method used by Fulton for Chow groups: one relies on the deformation to the normal cone to reduce to the case of a divisor, which is handled separately.

2.1. **Twists of oriented cohomology theories.** We recall from [10, § 4.1.8–9 & § 7.4.2] the construction of the twisting of an oriented cohomology theory on $\mathbf{Sm}_k$. Let A^* be such a theory and define $\lambda_{(\tau)}(u) = \sum_{i\geq 0} \tau_i u^{i+1} \in A^*(k)[[u]]$, where $\tau = (\tau_i) \in \prod_{i=0}^{\infty} A^{-i}(k)$ and $\tau_0 = 1$. This last condition ensures that $\lambda_{(\tau)}$ admits an inverse $\lambda_{(\tau)}^{-1}$.

Definition 2.5. The *inverse Todd class operator* of a line bundle $L \to X$ is defined to be the operator on $A^*(X)$ given by the infinite sum

$$\widetilde{Td}_\tau^{-1}(L) = \sum_{i=0}^{\infty} \tilde{c}_1(L)^i \tau_i.$$

In [10, Proposition 4.1.20], Levine and Morel showed that this definition can be extended uniquely to all vector bundles by imposing that for all exact sequences $0 \to E' \to E \to E'' \to 0$ over X one has $\widetilde{Td}_\tau^{-1}(E) = \widetilde{Td}_\tau^{-1}(E') \circ \widetilde{Td}_\tau^{-1}(E'')$. Thus, it naturally extends to a map

$$\widetilde{Td}_\tau^{-1} : K^0(X) \to \mathrm{Aut}(A^*(X)).$$

In fact, on an oriented cohomology theory on $\mathbf{Sm}_k$, $\widetilde{Td}_\tau^{-1}(E)$ is simply the multiplication by the element $Td_\tau^{-1}(E) := \widetilde{Td}_\tau^{-1}(E)(1_X) \in A^*(X)$, called the *inverse Todd class* of E. For any smooth equidimensional $Y \xrightarrow{f} X$, it is shown that $Td_\tau^{-1}(f^*E) = f^* Td_\tau^{-1}(E)$.

Suppose X, Y are in $\mathbf{Sm}_k$. Then, any $f : Y \to X$ is an l.c.i. morphism. Let $f = q \circ i$ be a factorization such that $i : Y \to P$ is a regular embedding and $q : P \to X$ is smooth. Letting $\mathcal{I}$ be the ideal sheaf of Y in P, we define the normal bundle N_i to be the bundle over Y whose dual has $\mathcal{I}/\mathcal{I}^2$ as the sheaf of sections. Set $N_f \in K^0(Y)$ to be the class $[N_i] - [i^*T_q]$, where T_q is the relative tangent bundle associated to q. For any τ as above, one may construct an oriented cohomology theory on $\mathbf{Sm}_k$, denoted $A^*_{(\tau)}$, by twisting the first Chern classes and the pull-back maps. If f^* and c_1 are respectively the pull-backs and the first Chern class respectively in A^*, then as groups $A^*_{(\tau)}(X) = A^*(X)$ and in $A^*_{(\tau)}$ one has:

- $f^*_{(\tau)} = Td_\tau^{-1}(N_f) \cdot f^*$, where Td_τ^{-1} is the inverse Todd class;
- for any line bundle L over X, the first Chern class of L in $A^*_{(\tau)}$ is $c_1^{(\tau)}(L) = \lambda_{(\tau)}(c_1(L))$;
- if $\cdot$ denotes the product in $A^*(X)$ and $\cdot_\tau$ denotes the product in $A^*_{(\tau)}(X)$, then $x \cdot_\tau y = Td_\tau^{-1}(N_{\delta_X}) \cdot x \cdot y$, for any $x, y \in A^*(X)$, where $\delta_X : X \to X \times X$ is the diagonal morphism.

Notice that the push-forward maps f_* are unchanged and that the modification of the first Chern classes affects the formal group law which becomes $F_A^{(\tau)}(u,v) = \lambda_{(\tau)}(F_A(\lambda_{(\tau)}^{-1}(u), \lambda_{(\tau)}^{-1}(v)))$.

If $f = q \circ i$ is a factorization as before, then note that P is smooth over k, since X is smooth and q is smooth. Thus, considering $\begin{array}{ccc} Y & \xrightarrow{i} & P \\ & \searrow \ \swarrow & \\ & \mathrm{Spec}\, k & \end{array}$, we get by [6, B.7.2.], the exact sequence $0 \longrightarrow T_Y \longrightarrow i^*T_P \longrightarrow N_i \longrightarrow 0$. Thus, in $K^0(Y)$ one has $[T_Y] = [i^*T_P] - [N_i]$ and since q is smooth, $[T_q] = [T_P] - [q^*T_X]$. Hence,

$$N_f = [i^*T_P] - [T_Y] - i^*([T_P] - [q^*T_X]) = [f^*T_X] - [T_Y].$$

Let us now consider more in detail the situation in the special case of abelian varieties: the tangent bundles T_X and T_Y become trivial. Thus, f^*T_X is trivial as well. It follows from the properties of $\widetilde{Td}_\tau^{-1}$ that

$$\begin{aligned} f^*_{(\tau)} = Td_\tau^{-1}(N_f) \cdot f^* &= Td_\tau^{-1}(f^*T_X) \cdot Td_\tau^{-1}(-T_Y) \cdot f^* \\ &= 1_{A^*(Y)} \cdot (Td_\tau^{-1}(T_Y))^{-1} \cdot f^* = 1_{A^*(Y)} \cdot f^* = f^*. \end{aligned} \tag{2.1}$$

Note that, if X is an abelian variety, then $Td_\tau^{-1}(N_{\delta_X}) = 1_{A^*(X)}$ since $X \times X$ is also an abelian variety and δ_X is an l.c.i. morphism. Thus, we obtain the following:

Lemma 2.6. *For an abelian variety X, there is a ring isomorphism $i_A^\tau : A^*_{(\tau)}(X) \xrightarrow{\sim} A^*(X)$.*

Proof. Since $A^*(X)$ and $A^*_{(\tau)}(X)$ coincide as groups, we only need to verify that the twisting does not affect the multiplicative structure. Let $\cdot$ denote the product in $A^*(X)$ and $\cdot_\tau$ denote the product in $A^*_{(\tau)}(X)$. Then, for $\alpha, \beta \in A^*(X)$, $\alpha \cdot_\tau \beta = Td_\tau^{-1}(N_{\delta_X}) \cdot \alpha \cdot \beta = \alpha \cdot \beta$. Thus the map $A^*_{(\tau)}(X) \to A^*(X)$ identifying the two groups is a ring isomorphism as well. □

Remark 2.7. Note that, even though i^τ_A is an isomorphism, it is not true in general that $i^\tau_A([Y \to X]_{A_{(\tau)}}) = [Y \to X]_A$. However, if $\pi : X \to \operatorname{Spec} k$ is the structure morphism of an abelian variety, then the argument in (2.1) shows that $i^\tau_A(1_X^{A_{(\tau)}}) = i^\tau_A(\pi^*_{(\tau)}(1_k)) = \pi^*(1_k) = 1_X^A$ since the tangent bundle on $\operatorname{Spec} k$ is also trivial.

Now we would like to highlight two applications of the twisting construction, which will be used in the definition of the generalized Fourier-Mukai transformation. As Levine and Morel point out in [10, Remark 4.2.11], the Chern character $ch : K^0 \to CH^*_{\mathbb{Q}}$ can be recovered by making use of Theorem 2.4. By an appropriate twisting, $CH^*_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ can be made into a multiplicative theory and the theorem then provides a morphism of oriented cohomology theories whose degree 0 component is precisely ch. For our purposes we will need to consider a generalization of this morphism for all ordinary theories with rational coefficients. For any such theory $A^*_{\mathbb{Q}}$ we define ch_A to be the composition

$$ch_A := \nu^{CH_{\mathbb{Q}}}_{A_{\mathbb{Q}}} \circ ch,$$

where $\nu^{CH_{\mathbb{Q}}}_{A_Q}$ is obtained from the morphism ν^{CH}_A (arising from Theorem 2.3) by enlarging the coefficient ring. Since $\nu^{CH_{\mathbb{Q}}}_{A_{\mathbb{Q}}}$ is a morphism of oriented cohomology theories, it immediately follows that for all $X \in \mathbf{Sm}_k$ one has that $ch_A(X) : K^0(X) \to A^*_{\mathbb{Q}}(X)$ is a ring homomorphism and that ch_A commutes with pushforward maps.

Another application of the twisting construction is to associate to any theory with rational coefficients an ordinary one. In fact, [10, Lemma 4.1.29] shows that there exists $\eta = (\eta_i) \in \prod_{i=0}^\infty A^*_{\mathbb{Q}}(k)$ with $\eta_0 = 1$, such that $\log_A(u) = \lambda_{(\eta)}$ is the logarithm of the formal group law F_A, *i.e.*

$$\log_A(F_A(u, v)) = \log_A(u) + \log_A(v).$$

The resulting theory $A^*_{ad} := A^*_{\mathbb{Q}\,(\eta)}$ is therefore ordinary and its first Chern classes are given by the formula $c_1^{A_{ad}}(L) = \log_A(c_1^A(L))$. For simplicity the isomorphism i^η_A will be denoted by i^{ad}_A.

It turns out to be useful to find the η_i's in terms of the coefficients of F_A.

Lemma 2.8. *If* $F_A(u, v) = u + v + \sum_{m \geq n \geq 1} a_{m,n}(u^m v^n + u^n v^m)$*, then* $\eta_i \in A^{-i}_{\mathbb{Q}}(k)$ *and moreover* $(i + 1)\eta_i \in A^{-i}(k)$.

Proof. We proceed by comparing the coefficients of certain terms on both sides of the equality $\log_A(F_A(u, v)) = \log_A(u) + \log_A(v)$. By looking at the coefficient of uv, we get $a_{1,1} + 2\eta_1 = 0$, so that $\eta_1 = -\frac{a_{1,1}}{2}$. Now consider the term uv^i for some $i \geq 1$. The only way for uv^i to be a product of k terms is $uv^{i-k+1} \cdot \underbrace{v \cdots v}_{k-1}$ for $1 \leq k \leq i + 1$. Thus, the coefficient of uv^i in the left hand side is $\sum_{k=1}^{i+1} \binom{k}{k-1} \eta_{k-1} a_{i-k+1,1}$ where we have $a_{0,1} = 1$. Equating this sum to 0, we have $(i + 1)\eta_i = -\sum_{k=1}^{i} k\eta_{k-1} a_{i-k+1,1}$. Note that $a_{m,1} \in A^{-m}(k)$. Thus, by applying induction, we may conclude from the above that $(i + 1)\eta_i \in A^{-i}(k)$. □

3. Recollection of A-motives

In [14, § 5-6], for an oriented cohomology theory A^* on $\mathbf{Sm}_k$, Nenashev and Zainoulline constructed the A-motive of a smooth projective variety X over k, following the ideas of [11]. We briefly recall its construction.

3.1. **A-correspondences.** Let X and Y be smooth projective varieties over an algebraically closed field k of characteristic 0. We recall from [14] some facts about the category of *A-correspondences.* Given an oriented cohomology theory A^*, we define the category of A-correspondences, denoted Cor_A, as

- $Ob(Cor_A) := Ob(\mathbf{SmProj}_k)$;
- $\mathrm{Hom}_{Cor_A}(X,Y) := A^*(X \times Y)$;
- the composition of morphisms $\alpha \in A^*(X \times Y)$ and $\beta \in A^*(Y \times Z)$ is the correspondence
$$\beta \circ \alpha := (p_{XZ})_*(p_{XY}^*(\alpha) \cdot p_{YZ}^*(\beta)) \ \in A^*(X \times Z).$$
where p_{XZ}, p_{XY} and p_{YZ} are the respective projections from $X \times Y \times Z$.

There is a functor $c_A : \mathbf{SmProj}_k^{op} \to Cor_A$ given by

$$c_A(X) = X \text{ and } c_A(f) = (\Gamma_f)_*(1_{A(X)}) \in A^*(Y \times X)$$

for a morphism $f : X \to Y$, where $\Gamma_f : X \xrightarrow{(f,\mathrm{id})} Y \times X$ is the graph morphism. For $\alpha \in A^*(X \times Y)$, we have the transpose $\alpha^t := \iota^*(\alpha) \in A^*(Y \times X)$, where $\iota : Y \times X \to X \times Y$ is given by swapping the variables.

For a correspondence $\alpha \in \mathrm{Hom}_{Cor_A}(Y,X)$, we define its realization $\mathcal{R}_A(\alpha) : A^*(Y) \to A^*(X)$ as follows: we identify $A^*(Y)$ with $\mathrm{Hom}_{Cor_A}(pt,Y)$ and note that α defines a map $\mathrm{Hom}_{Cor_A}(pt,Y) \to \mathrm{Hom}_{Cor_A}(pt,X)$ given by composition with α. This defines the map $\mathcal{R}_A(\alpha)$ as

$$\beta \mapsto p_{X*}(\alpha \cdot p_Y^*\beta),$$

where p_X and p_Y are the respective projections to X and Y from $Y \times X$. When no confusion is likely to arise we will replace $\mathcal{R}_A$ by $\mathcal{R}$. Note that the projection formula for the oriented cohomology theory A^* implies that

$$\mathcal{R}(c_A(f)) = f^* \text{ and } \mathcal{R}(c_A(f)^t) = f_*, \tag{3.1}$$

so that the functor $A^* : \mathbf{SmProj}_k^{op} \to \mathbf{Ab}^{\mathbb{Z}}$ factors through Cor_A.

If $\alpha \in A^*(X \times Y)$ and $\beta \in A^*(Y \times Z)$, it follows from the definition that

$$\mathcal{R}(\beta) \circ \mathcal{R}(\alpha) = \mathcal{R}(\beta \circ \alpha) = \mathcal{R}\big((p_{XZ})_*(p_{XY}^*(\alpha) \cdot p_{YZ}^*(\beta))\big), \tag{3.2}$$

Also, using the Projection formula, one has

$$c_A(f) \circ \alpha = (\mathrm{id}_Z \times f)^*(\alpha) \quad \text{and} \quad \beta \circ c_A(f) = (f \times \mathrm{id}_Z)_*(\beta) \tag{3.3}$$

for $f : X \to Y$, $\alpha \in A^*(Z \times Y)$ and $\beta \in A^*(X \times Z)$. Applying transpose, we also get that for $\gamma \in A^*(Y \times Z)$ and $\delta \in A^*(Z \times X)$,

$$\gamma \circ c_A(f)^t = (f \times \mathrm{id}_Z)^*(\gamma) \quad \text{and} \quad c_A(f)^t \circ \delta = (\mathrm{id}_Z \times f)_*(\delta). \tag{3.4}$$

The grading on A^* induces a grading on Hom_{Cor_A} which makes it into a graded algebra under composition. It is given as

$$\mathrm{Hom}^n_{Cor_A}(X,Y) := \oplus_i A^{n+d_i}(X_i \times Y),$$

where the X_i's are the irreducible components of X and $d_i = \mathrm{dim} X_i$. It is worth stressing that Cor_A forms an additive category by defining $X \oplus Y = X \coprod Y$.

3.2. **A-motives.**

Definition 3.1. Consider the category Cor_A^0 with the same objects as Cor_A and $\mathrm{Hom}_{Cor_A^0}(X,Y) := \mathrm{Hom}^0_{Cor_A}(X,Y)$. The pseudo-abelian completion of Cor_A^0 is called the *category of effective A-motives*, denoted by $\mathcal{M}_A^{\mathrm{eff}}$. This means that the objects in $\mathcal{M}_A^{\mathrm{eff}}$ are pairs (X,p) where $X \in Ob(Cor_A)$ and $p \in \mathrm{Hom}_{Cor_A^0}(X,X)$ is a projector (that is, $p \circ p = p$) and that the morphisms are given by

$$\mathrm{Hom}_{\mathcal{M}_A^{\mathrm{eff}}}((X,p),(Y,q)) = \frac{\{\alpha \in \mathrm{Hom}_{Cor_A^0}(X,Y) | \alpha \circ p = q \circ \alpha\}}{\{\alpha \in \mathrm{Hom}_{Cor_A^0}(X,Y) | \alpha \circ p = q \circ \alpha = 0\}}.$$

The *category of A-motives*, denoted by $\mathcal{M}_A$, has as objects triplets (X, p, m) where (X, p) is an object in $\mathcal{M}_A^{\text{eff}}$ and $m \in \mathbb{Z}$. The morphisms are defined as:

$$\mathrm{Hom}_{\mathcal{M}_A}((X,p,m),(Y,q,n)) = \frac{\{\alpha \in \mathrm{Hom}^{n-m}_{Cor_A}(X,Y) | \alpha \circ p = q \circ \alpha\}}{\{\alpha \in \mathrm{Hom}^{n-m}_{Cor_A}(X,Y) | \alpha \circ p = q \circ \alpha = 0\}}.$$

Note that this means $\mathrm{id}_{(X,p,0)} = \mathrm{id}_X = p \in \mathrm{Hom}_{\mathcal{M}_A}((X,p,0),(X,p,0))$. We abuse notation to write id_X to mean $c_A(\mathrm{id}_X) \in \mathrm{Hom}^0_{Cor_A}(X,X)$. The motive $(X, \mathrm{id}_X, 0)$ is called the motive of X and is denoted by $h_A(X)$. The additive structure of Cor_A induces a direct sum in the category $\mathcal{M}_A$.

4. Fourier-Mukai operator

4.1. **Notation for Abelian varieties.** From now on, unless stated otherwise, $X \xrightarrow{\pi_X} \operatorname{Spec} k$ will be an abelian variety over k of dimension g. Its dual abelian variety will be denoted $\hat{X}$ and $\mathfrak{P} \to X \times \hat{X}$ will represent the normalized Poincaré bundle with $\mathcal{P}$ being its class in the Grothendieck ring of vector bundles. Here, "normalized" means that $i^*\mathfrak{P}$ and $\hat{i}^*\mathfrak{P}$ are trivial, where $i : \{0\} \times \hat{X} \to X \times \hat{X}$ and $\hat{i} : X \times \{\hat{0}\} \to X \times \hat{X}$ are inclusions. By definition X comes equipped with a group operation $\mu : X \times X \to X$, whose associated inverse morphism is denoted by σ_X. For any integer m we will write $\mathbf{m} : X \to X$ to represent the morphism of multiplication by m with respect to the operation μ. As a general principle, we will add a ˆ to denote the corresponding notion for the dual abelian variety.

Finally, let us recall that for any oriented cohomology theory A^* it is possible to define on $A^*(X)$ the so-called Pontryagin product. It will be denoted by $\star$ and it is defined as

$$x \star y := \mu_*(p_1^* x \cdot p_2^* y),$$

where p_1 and p_2 are the projections of $X \times X$ onto the first and second factor, and $\cdot$ is the usual product on A^*.

4.2. **The definition of $\mathcal{F}$.** For any oriented cohomology theory A^* we wish to define an operator $\mathcal{F}^A : A^*_{\mathbb{Q}}(X) \to A^*_{\mathbb{Q}}(\hat{X})$ which has the usual properties of the Fourier-Mukai transformation on Chow rings or K-theory (see [2]). The key observation that allows one to extend the definition given by Beauville is that the Fourier-Mukai transformation for $CH^*_{\mathbb{Q}}$ can be restated in terms of correspondences: $\mathcal{F}^{CH} = \mathcal{R}(ch(\mathcal{P}))$. Hence one is left with the task of indentifying the correct analogue of $ch(\mathcal{P})$ in $A^*(X \times \hat{X})$. Our proposal is to make use of the twisting construction to reduce to the case of an ordinary theory and then use the universality of CH^* among such theories. This leads to the definition of $\mathcal{C}_A : K^0(X) \to A^*_{\mathbb{Q}}(X)$ as

$$\mathcal{C}_A := i^{ad}_{A_{\mathbb{Q}}} \circ ch_A.$$

It follows directly from the definition that for any line bundle L on X we have

$$\mathcal{C}_A([L]) = \exp(\log_A(c_1^A(L))). \tag{4.1}$$

Proposition 4.1. *Let A^* be any oriented cohomology theory. Then,*

(1) *$\mathcal{C}_A$ is a ring homomorphism;*
(2) *$\mathcal{C}_A$ commutes with pullbacks of morphisms between abelian varieties;*
(3) *$\mathcal{C}_A$ commutes with pushforwards of morphisms between abelian varieties.*

Proof. The first statement is obvious since $\mathcal{C}_A$ is the composition of two rings homomorphisms. For the second and the third stament one first recalls that by definition $\mathcal{C}_A = i^{ad}_{A_{\mathbb{Q}}} \circ \nu^{CH_{\mathbb{Q}}}_{A_{\mathbb{Q}}} \circ ch$, so it suffices to prove that the statements hold for each of the morphisms separately. (2) holds for $i^{ad}_{A_{\mathbb{Q}}}$ in view of (2.1), while (3) follows since the twisting does not modify the pushforwards. On the other hand, $\nu^{CH}_{A_{\mathbb{Q}}}$ is a morphism of oriented cohomology theories and hence by definition commutes with both pushforwards and pullbacks. Finally both statements are verified for the Chern character as well, since we are dealing with abelian varieties, see [6, §15.1 and Theorem 15.2]. □

Definition 4.2. Let A^* be any oriented cohomology theory. We define the Fourier-Mukai operator $\mathcal{F}^A : A^*_{\mathbb{Q}}(X) \to A^*_{\mathbb{Q}}(\hat{X})$ to be $\mathcal{F}^A := \mathcal{R}(\mathcal{C}_A(\mathcal{P}))$. The dual operator $\hat{\mathcal{F}}^A$ is defined to be $\mathcal{R}((\mathcal{C}_A(\mathcal{P}))^t)$. When there is no confusion, we will denote $\mathcal{F}^A$ and $\hat{\mathcal{F}}^A$ by $\mathcal{F}$ and $\hat{\mathcal{F}}$ respectively.

The operators $\mathcal{F}$ and $\hat{\mathcal{F}}$ defined above satisfy the following:

Theorem 4.3. *We have*

$$\hat{\mathcal{F}} \circ \mathcal{F} = (-1)^g \sigma_X^*$$
$$\mathcal{F} \circ \hat{\mathcal{F}} = (-1)^g \sigma_{\hat{X}}^*,$$

where $\sigma_X : X \to X$ is the multiplication by (-1) in the abelian variety X.

Proof. It follows from (3.2) that $\hat{\mathcal{F}} \circ \mathcal{F} = \mathcal{R}\big(q_{13*}(q_{12}^*(\mathcal{C}_A(\mathcal{P})) \cdot q_{23}^*(\iota^*\mathcal{C}_A(\mathcal{P})))\big)$ where q_{ij} are the projections from $X \times \hat{X} \times X$. We may rewrite this as

$$\hat{\mathcal{F}} \circ \mathcal{F} = \mathcal{R}\big(p_{12*}(p_{13}^*(\mathcal{C}_A(\mathcal{P})) \cdot p_{23}^*(\mathcal{C}_A(\mathcal{P})))\big)$$

where p_{ij} are the respective projections from $X \times X \times \hat{X}$. It follows from Proposition 4.1 that $p_{12*}(p_{13}^*(\mathcal{C}_A(\mathcal{P})) \cdot p_{23}^*(\mathcal{C}_A(\mathcal{P}))) = \mathcal{C}_A(p_{12*}[p_{13}^*\mathfrak{P} \otimes p_{23}^*\mathfrak{P}])$. But, by the Theorem of the Cube ([13, § 6]), $p_{13}^*\mathfrak{P} \otimes p_{23}^*\mathfrak{P} \simeq (\mu \times \mathrm{id}_{\hat{X}})^*\mathfrak{P}$. Indeed, by the Theorem, it is enough to verify this on $\{0\} \times X \times \hat{X}$, $X \times \{0\} \times \hat{X}$ and $X \times X \times \{\hat{0}\}$, where it follows from the properties of the Poincaré bundle. Also, since μ and $p_1 : X \times \hat{X} \to X$ are transverse morphisms, we have $p_{12*}(\mu \times \mathrm{id}_{\hat{X}})^*\mathcal{P} = \mu^* p_{1*}\mathcal{P}$. But, in $K^0(X \times \hat{X})$, $p_{1*}\mathcal{P} = \sum_i (-1)^i [\mathbf{R}^i p_{1*}\mathfrak{P}]$. From [13, § 13] we have that $\mathbf{R}^i p_{1*}\mathfrak{P} = 0$ for $i \neq g$ and $\mathbf{R}^g p_{1*}\mathfrak{P} = k(0)$ where $k(0)$ is the sky-scrapper sheaf at the point $0 \in X$. Note that $\mu^* k(0) \xrightarrow{\sim} \mathcal{O}_{\Gamma_{\sigma_X}(X)}$ in $K^0(X)$. Since Γ_{σ_X} is a closed immersion, $\mathcal{O}_{\Gamma_{\sigma_X}(X)} \xrightarrow{\sim} \Gamma_{\sigma_X *} 1_X^K$. Putting all these together, $\mathcal{C}_A(\mu^* p_{1*}\mathcal{P}) = \mathcal{C}_A((-1)^g \Gamma_{\sigma_X *} 1_X^K) = (-1)^g \Gamma_{\sigma_X *}\mathcal{C}_A(1_X^K)$. Finally, in view of Remark 2.7 we have $\mathcal{C}_A(1_X^K) = 1_X^A$, therefore $\hat{\mathcal{F}} \circ \mathcal{F} = \mathcal{R}((-1)^g c(\sigma_X)) = (-1)^g \mathcal{R}(c(\sigma_X)) = (-1)^g \sigma_X^*$. The other part is proved analogously. □

5. Properties of the Fourier-Mukai operator

We will need the following lemma:

Lemma 5.1. *Let $f : X \to Y$ be a finite surjective morphism between abelian varieties X and Y. Then, for any $x \in A_{\mathbb{Q}}^*(X)$,*

$$f_* f^* x = (\deg f) \cdot x \quad \in A_{\mathbb{Q}}^*(X),$$

where deg *is the degree of a morphism.*

Proof. By the projection formula, we get $f_* f^* x = f_*(f^* x \cdot 1_X^A) = x \cdot f_*(1_X^A)$. Note that by Remark 2.7, $i_A^{ad}(f_* 1_X^{A_{ad}}) = f_* 1_X^A$. But, $f_* 1_X^{A_{ad}} = \nu_{A_{ad}}^{\mathrm{CH}}(f_* 1_X^{\mathrm{CH}}) = \nu_{A_{ad}}^{\mathrm{CH}}((\deg f) \cdot 1_X^{\mathrm{CH}}) = (\deg f) \cdot 1_X^{A_{ad}}$, which completes the proof. □

The Fourier-Mukai operator on $A_{\mathbb{Q}}^*$ of abelian varieties satisfies the following properties:

Proposition 5.2. (1) *For $x, y \in A_{\mathbb{Q}}^*(X)$, we have*

$$\mathcal{F}^A(x \star y) = \mathcal{F}^A(x)\mathcal{F}^A(y) \quad \textit{and} \quad \mathcal{F}^A(xy) = (-1)^g \mathcal{F}^A(x) \star \mathcal{F}^A(y).$$

(2) *Let $f : X \to Y$ be an isogeny of abelian varieties, and $\hat{f} : \hat{Y} \to \hat{X}$ be the dual isogeny. Then one has*

$$\mathcal{F}_Y^A \circ f_* = \hat{f}^* \circ \mathcal{F}_X^A \quad \textit{and} \quad \mathcal{F}_X^A \circ f^* = \hat{f}_* \circ \mathcal{F}_Y^A.$$

(3) *Let $x \in A_{\mathbb{Q}}^p(X)$. Write $\mathcal{F}^A(x) = \sum_q y_q$, where $y_q \in A_{\mathbb{Q}}^q(\hat{X})$. Then, for $n \in \mathbb{Z}$, one has*

$$\mathbf{n}^* y_q = n^{g-p+q} y_q,$$

where $\mathbf{n}$ denotes the multiplication by n on $\hat{X}$.

Proof. **(1):** First, we want to prove $\mathcal{F}^A(x\star y)=\mathcal{F}^A(x)\mathcal{F}^A(y)$. Note that we have the following commutative diagram.

$$\begin{array}{ccccc}
X\times X\times\hat{X} & \xrightarrow{\mu\times\mathrm{id}} & X\times\hat{X} & \xrightarrow{p_{\hat{X}}} & \hat{X}\\
\downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_X} & & \\
X\times X & \xrightarrow{\mu} & X & & \\
\downarrow{\scriptstyle p_1} & \searrow^{p_2} & & & \\
X & & X & &
\end{array}$$

Let p_{12}, p_{23} and p_{13} be the respective projections of $X\times X\times\hat{X}$. Also, denote by π_i, the projection of $X\times X\times\hat{X}$ to the i-th factor, for $i=1,2,3$. By definition we have that $x\star y=\mu_*(p_1^*x\cdot p_2^*y)$, so we obtain the following sequence of equalities.

$$\begin{aligned}
\mathcal{F}^A(x\star y)&=p_{\hat{X}*}\big(\mathcal{C}_A(\mathcal{P})\cdot p_X^*\mu_*(p_1^*x\cdot p_2^*y)\big)=p_{\hat{X}*}\big(\mathcal{C}_A(\mathcal{P}\cdot(\mu\times\mathrm{id}_X)_*p_{12}^*(p_1^*x\cdot p_2^*y))\big)\\
&=\pi_{3*}\big(p_{13}^*\mathcal{C}_A(\mathcal{P})\cdot p_{23}^*\mathcal{C}_A(\mathcal{P})\cdot\pi_1^*x\cdot\pi_2^*y\big)\quad[\text{Since }(\mu\times\mathrm{id}_X)^*\mathfrak{P}=p_{13}^*\mathfrak{P}\otimes p_{23}^*\mathfrak{P}]\\
&=\pi_{3*}\big(p_{13}^*(\mathcal{C}_A(\mathcal{P})\cdot p_X^*x)\cdot p_{23}^*(\mathcal{C}_A(\mathcal{P})\cdot p_X^*y)\big)=p_{\hat{X}*}p_{13*}\big(p_{13}^*(\mathcal{C}_A(\mathcal{P})\cdot p_X^*x)\cdot p_{23}^*(\mathcal{C}_A(\mathcal{P})\cdot p_X^*y)\big)\\
&=p_{\hat{X}*}\big((\mathcal{C}_A(\mathcal{P})\cdot p_X^*x)\cdot p_{13*}p_{23}^*(\mathcal{C}_A(\mathcal{P})\cdot p_X^*y)\big)=p_{\hat{X}*}\big((\mathcal{C}_A(\mathcal{P})\cdot p_X^*x)\cdot p_{\hat{X}}^*p_{\hat{X}*}(\mathcal{C}_A(\mathcal{P})\cdot p_X^*y)\big)
\end{aligned}$$

This is true since p_{13} and p_{23} are transverse morphisms. Thus, applying the Projection formula yields

$$\mathcal{F}^A(x\star y)=p_{\hat{X}*}\big(\mathcal{C}_A(\mathcal{P})\cdot p_X^*x\big)\cdot p_{\hat{X}*}\big(\mathcal{C}_A(\mathcal{P})\cdot p_X^*y\big)=\mathcal{R}(\mathcal{C}_A(\mathcal{P})(x))\cdot\mathcal{R}(\mathcal{C}_A(\mathcal{P})(y))=\mathcal{F}^A(x)\cdot\mathcal{F}^A(y).$$

Similarly, we can also show that for $x',y'\in A^*_{\mathbb{Q}}(\hat{X})$, we have $\hat{\mathcal{F}}^A(x'\star y')=\hat{\mathcal{F}}^A(x')\cdot\hat{\mathcal{F}}^A(y')$. Note that, in view of Theorem 4.3, this implies the other statement. Indeed,

$$\begin{aligned}
\sigma_{\hat{X}}^*(\mathcal{F}^A(x)\star\mathcal{F}^A(y))&=(-1)^g\mathcal{F}^A\circ\hat{\mathcal{F}}^A(\mathcal{F}^A(x)\star\mathcal{F}^A(y))=(-1)^g\mathcal{F}^A\big(\hat{\mathcal{F}}^A(\mathcal{F}^A(x))\cdot\hat{\mathcal{F}}^A(\mathcal{F}^A(y))\big)\\
&=(-1)^g\mathcal{F}^A\big(\sigma_X^*(xy)\big)=\mathcal{F}^A\big(\hat{\mathcal{F}}^A\circ\mathcal{F}^A(xy)\big)=(-1)^g\sigma_{\hat{X}}^*\mathcal{F}^A(xy)
\end{aligned}$$

Since $\sigma_{\hat{X}}\circ\sigma_{\hat{X}}=\mathrm{id}_{\hat{X}}$, the result follows.

(2): The combined use of (3.1), (3.2), (3.4) and (3.3) shows that

$$\begin{aligned}
\mathcal{F}_Y^A\circ f_*&=\mathcal{R}(\mathcal{C}_A(\mathcal{P}_Y))\circ\mathcal{R}(c(f)^t)=\mathcal{R}(\mathcal{C}_A(\mathcal{P}_Y)\circ c(f)^t)=\mathcal{R}((f\times\mathrm{id}_{\hat{Y}})^*\mathcal{C}_A(\mathcal{P}_Y))\text{ and}\\
\hat{f}^*\circ\mathcal{F}_X^A&=\mathcal{R}(c(\hat{f}))\circ\mathcal{R}(\mathcal{C}_A(\mathcal{P}_X))=\mathcal{R}(c(\hat{f})\circ\mathcal{C}_A(\mathcal{P}_X))=\mathcal{R}((\mathrm{id}_X\times\hat{f})^*\mathcal{C}_A(\mathcal{P}_X)).
\end{aligned}$$

But, by definition of $\hat{f}$, $(\mathrm{id}_X\times\hat{f})^*\mathcal{P}_X=(f\times\mathrm{id}_{\hat{Y}})^*\mathcal{P}_Y$. Similarly, the second assertion will follow if we show that

$$(f\times\mathrm{id}_{\hat{X}})_*\mathcal{C}_A(\mathcal{P}_X)=(\mathrm{id}_Y\times\hat{f})_*\mathcal{C}_A(\mathcal{P}_Y).$$

To see this, note that the transversality of the square

$$\begin{array}{ccc}
X\times\hat{Y} & \xrightarrow{\mathrm{id}_X\times\hat{f}} & X\times\hat{X}\\
\downarrow{\scriptstyle f\times\mathrm{id}_{\hat{Y}}} & & \downarrow{\scriptstyle f\times\mathrm{id}_{\hat{X}}}\\
Y\times\hat{Y} & \xrightarrow{\mathrm{id}_Y\times\hat{f}} & Y\times\hat{X}
\end{array}$$

gives us in $A^*_{\mathbb{Q}}(X)$ that

$$\begin{aligned}
(\mathrm{id}_Y\times\hat{f})^*(f\times\mathrm{id}_{\hat{X}})_*\mathcal{C}_A(\mathcal{P}_X)&=(f\times\mathrm{id}_{\hat{Y}})_*(\mathrm{id}_X\times\hat{f})^*\mathcal{C}_A(\mathcal{P}_X)=(f\times\mathrm{id}_{\hat{Y}})_*(f\times\mathrm{id}_{\hat{Y}})^*\mathcal{C}_A(\mathcal{P}_Y)\\
&=(\deg f)\mathcal{C}_A(\mathcal{P}_Y)\quad[\text{By Lemma 5.1, since }\deg(f\times\mathrm{id}_{\hat{Y}})=\deg f].
\end{aligned}$$

To finish the proof it suffices to apply $(\mathrm{id}_Y\times\hat{f})_*$ to both sides and use Lemma 5.1.

(3): Consider the endomorphism $(1_X,\mathbf{n})$ of $X\times\hat{X}$. We first note that the Theorem of the Squares ([13, II.6 - Corollary 4]) implies that $(1_X,\mathbf{n})^*\mathfrak{P}=\mathfrak{P}^{\otimes n}$. Indeed, as the bundle is rigidified along zero sections, the theorem gives us

$$(1_X,\mathbf{n})^*\mathfrak{P}=(1_X,(\mathbf{n-1}))^*\mathfrak{P}\otimes(1_X,(\mathbf{n-1}))^*\mathfrak{P}\otimes(1_X,(\mathbf{n-2}))^*\mathfrak{P}^{-1}$$

and we have the above by induction.

Since $\mathcal{C}_A(\mathcal{P}) = \exp(\log_A(c_1^A(\mathfrak{P}))) = \exp(c_1^{A_{ad}}(\mathfrak{P}))$, one gets

$$(1_X, \mathbf{n})^*\mathcal{C}_A(\mathcal{P}) = \exp(c_1^{A_{ad}}(\mathfrak{P}^{\otimes n})) = \exp(nc_1^{A_{ad}}(\mathfrak{P})) = \sum_{i=0}^{2g} \frac{n^i}{i!}(c_1^{A_{ad}}(\mathfrak{P}))^i.$$

Thus, we have

$$\begin{aligned}\mathbf{n}^*\mathcal{F}^A(x) = \mathbf{n}^* p_{\hat{X}*}\big(p_X^* x \cdot \mathcal{C}_A(\mathcal{P})\big) &= p_{\hat{X}*}\big((1_X,\mathbf{n})^* p_X^* x \cdot (1_X,\mathbf{n})^*\mathcal{C}_A(\mathcal{P})\big)\\ &= p_{\hat{X}*}\big(p_X^* x \cdot \sum_{i=0}^{2g} \frac{n^i(c_1^{A_{ad}}(\mathfrak{P}))^i}{i!}\big) = \sum_{i=0}^{2g} n^i p_{\hat{X}*}\big(p_X^* x \cdot \frac{(c_1^{A_{ad}}(\mathfrak{P}))^i}{i!}\big)\end{aligned}$$

Note that $p_{\hat{X}*}\big(p_X^* x \cdot \frac{(c_1^{A_{ad}}(\mathfrak{P}))^i}{i!}\big) = y_{p+i-g}$. It now suffices to set $q = p + i - g$ to obtain

$$\mathbf{n}^*\mathcal{F}^A(x) = \sum_{q=p-g}^{p+g} n^{g-p+q} y_q,$$

which gives the desired equality. □

6. Beauville decomposition for oriented cohomology theories

We follow the ideas in [3] to give a decomposition of $A^*_{\mathbb{Q}}(X)$ into eigenspaces of $\mathbf{n}^*$ using the Fourier-Mukai operator defined in § 4. For $s \in \mathbb{Z}$, let ${A_{\mathbb{Q}}}_s^p(X)$ denote the sub-group

$${A_{\mathbb{Q}}}_s^p(X) := \big\{x \in A^p_{\mathbb{Q}}(X) \,|\, \forall\, n \in \mathbb{Z},\ \mathbf{n}^* x = n^{2p-s}x\big\}.$$

Proposition 6.1. *Let $x \in A^p_{\mathbb{Q}}(X)$, and m be any integer other than 0, 1 or −1. The following conditions are equivalent:*

(1) $\mathcal{F}^A(x) \in A_{\mathbb{Q}}^{g-p+s}(\hat{X})$;
(2) $x \in {A_{\mathbb{Q}}}_s^p(X)$;
(3) $\mathbf{m}^* x = m^{2p-s}x$;
(4) $\mathbf{m}_* x = m^{2g-2p+s}x$;
(5) $\mathcal{F}^A(x) \in {A_{\mathbb{Q}}}_s^{g-p+s}(\hat{X})$.

Proof. **(1)⇒(2):** Let $y = (-1)^g(\sigma_{\hat{X}})^*\mathcal{F}^A(x)$ and let $\hat{\mathcal{F}}^A(y) = \sum_q x_q$ with $x_q \in A^q_{\mathbb{Q}}(X)$. Then, Proposition 5.2, part (3) gives us $\mathbf{n}^* x_q = n^{g-(g-p+s)+q}x_q = n^{p+q-s}x_q$. But, by Proposition 5.2, part (2) and Theorem 4.3, we get

$$\begin{aligned}\hat{\mathcal{F}}^A(y) &= (-1)^g\hat{\mathcal{F}}^A \circ (\sigma_{\hat{X}})^* \circ \mathcal{F}^A(x) = (-1)^g(\sigma_X)_* \circ \hat{\mathcal{F}}^A \circ \mathcal{F}^A(x)\\ &= (\sigma_X)_*(\sigma_X)^*(x) = x, \quad \text{[By Lemma 5.1, since } \deg(\sigma_X) = 1.]\end{aligned}$$

Then, $x = x_p$ and $\mathbf{n}^* x = n^{2p-s}x$, thus showing that $x \in {A_{\mathbb{Q}}}_s^p(X)$.

(2)⇒(3): This is by definition.

(3)⇒(4): Since $\deg \mathbf{m} = m^{2g}$, in view of Lemma 5.1 one has $\mathbf{m}_*\mathbf{m}^* x = m^{2g}x$.

(4)⇒(5): By Proposition 5.2, part (2), we get that

$$\mathbf{m}^*\mathcal{F}^A(x) = \mathcal{F}^A(\mathbf{m}_* x) = m^{2g-2p+s}\mathcal{F}^A(x) = m^{g-p+(g-p+s)}\mathcal{F}^A(x).$$

Since $m \neq 0, \pm 1$, this implies by Proposition 5.2, part (3), that $\mathcal{F}^A(x) \in A_{\mathbb{Q}}^{g-p+s}(\hat{X})$. Then, by definition, $\mathcal{F}^A(x) \in {A_{\mathbb{Q}}}_s^{g-p+s}(\hat{X})$.

(5)⇒(1): This is obvious. □

Theorem 6.2. *Let X be an abelian variety of dimension g over k. Then, we have*

$$A^p_{\mathbb{Q}}(X) = \bigoplus_{s=2p-2g}^{2p} {A_{\mathbb{Q}}}_s^p(X).$$

Proof. Let $x \in A^p_{\mathbb{Q}}(X)$ and let $y = \mathcal{F}^A(x)$. We can write $y = \sum_q y_q$, with $y_q \in A^q_{\mathbb{Q}}(\hat{X})$. By Proposition 5.2, part (3), $\mathbf{n}^* y_q = n^{g-p+q} y_q$ and hence $y_q \in A_{\mathbb{Q}p+q-g}^{\ q}(\hat{X})$. Then, Proposition 6.1 gives us that $\hat{\mathcal{F}}^A(y_q) \in A_{\mathbb{Q}p+q-g}^{\ p}(X)$. But, $(-1)^g(\sigma_X)^* x = \hat{\mathcal{F}}^A(y) = \sum_q \hat{\mathcal{F}}^A(y_q)$.

Since $\hat{\mathcal{F}}^A(y_q)$ has degree p and by definition $\hat{\mathcal{F}}^A(y_q) = \sum_{i\geq 0} p_{1*}\left(p_2^*(y_q)\dfrac{(c_1^{A_{ad}}(\mathfrak{P}))^i}{i!}\right)$, we get $\hat{\mathcal{F}}^A(y_q) = p_{1*}\left(p_2^*(y_q)\dfrac{(c_1^{A_{ad}}(\mathfrak{P}))^{p+g-q}}{(p+g-q)!}\right)$. Also, $y_q = p_{2*}\left(p_1^*(x)\dfrac{(c_1^{A_{ad}}(\mathfrak{P}))^{q+g-p}}{(q+g-p)!}\right)$ since y_q has degree q. Now we notice that $(c_1^{A_{ad}}(\mathfrak{P}))^i$ vanishes unless $0 \leq i \leq 2g$ and this implies that y_q and $\hat{\mathcal{F}}^A(y_q)$ are both nonzero only if $p - g \leq q \leq p + g$. Since $s + g - p = q$, one immediately obtains the bounds, *i.e.* $2p - 2g \leq s \leq 2p$. □

Remark 6.3. For $\Omega^*_{\mathbb{Q}}$ the previous bound can be improved by observing that, since $\dim X = g$, for all $x \in \Omega^p(X)$ one has $\mathcal{F}^{\Omega}(x) = \sum_{q\leq g} y_q$. This allows one to conclude that $2p-g \leq s \leq \mathrm{Min}(2p, p)$. The same observation holds for all theories which can be obtained from algebraic cobordism by imposing a formal group law, provided that the coefficient ring of definition has no elements of positive degree. For instance, this is the case for the Chow ring.

As an easy consequence of the definition of $A_{\mathbb{Q}s}^{\ p}(X)$ and Proposition 6.1, we get the following:

Proposition 6.4. (1) $\mathcal{F}^A(A_{\mathbb{Q}s}^{\ p}(X)) = A_{\mathbb{Q}s}^{\ g-p+s}(\hat{X})$.

(2) *If $f : X \to Y$ is a homomorphism of abelian varieties of relative dimension m, then $f^* A_{\mathbb{Q}s}^{\ p}(Y) \subset A_{\mathbb{Q}s}^{\ p}(X)$ and $f_* A_{\mathbb{Q}s}^{\ p}(X) \subset A_{\mathbb{Q}s}^{\ p+m}(Y)$.*

(3) *If $x \in A_{\mathbb{Q}s}^{\ p}(X)$, $y \in A_{\mathbb{Q}t}^{\ q}(X)$, then $xy \in A_{\mathbb{Q}s+t}^{\ p+q}(X)$ and $x \star y \in A_{\mathbb{Q}s+t}^{\ p+q-g}(X)$.*

Proof. (1) is immediate from Proposition 6.1.

(2) follows from the fact that $f \circ \mathbf{m} = \mathbf{m} \circ f$ and the equivalence of (3) and (4) in Proposition 6.1.

Finally, if $x \in A_{\mathbb{Q}s}^{\ p}(X)$ and $y \in A_{\mathbb{Q}t}^{\ q}(X)$, then by definition $xy \in A_{\mathbb{Q}s+t}^{\ p+q}(X)$. Also, note that $\mathcal{F}^A(x \star y) = \mathcal{F}^A(x)\mathcal{F}^A(y) \in A_{\mathbb{Q}s+t}^{\ 2g-p-q+s+t}(\hat{X})$. Applying $\hat{\mathcal{F}}^A$, we get $(\sigma_X)^*(x \star y) \in A_{\mathbb{Q}s+t}^{\ p+q-g}(X)$, which gives the result by part (2). □

6.1. **Application to K-theory.** Note that the Fourier-Mukai transformation that Beauville defined on K-theory in [2, § 1] does not lead to a decomposition of K^0 as there is no grading. However, Beauville's work on the Chow ring induces one via the usual Chern character ch. We now want to explicitly express this decomposition by using Theorem 6.2 for the oriented cohomology theory $K^0[\beta, \beta^{-1}]_{\mathbb{Q}}$ and consider its zero-component.

Applying the argument in the proof of Lemma 2.8 to $A^* = K^0[\beta, \beta^{-1}]^*$ readily gives us $\eta_i = \dfrac{\beta^i}{i+1}$, so that $\log_{K^0[\beta,\beta^{-1}]}(x) = \sum_{i=1}^{\infty} \dfrac{\beta^{i-1}}{i} x^i$ for $x \in K^0[\beta, \beta^{-1}]^*(X)$. Thus,

$$c_1^{K^0[\beta,\beta^{-1}]_{ad}}(\mathfrak{P}) = \log_{K^0[\beta,\beta^{-1}]}(\beta^{-1}(1 - [\mathfrak{P}^\vee])) = \beta^{-1} \sum_{i=1}^{\infty} \frac{(1 - [\mathfrak{P}^\vee])^i}{i} = \beta^{-1}\log([\mathfrak{P}^\vee])$$

where log is the usual logarithm series.

Now, pick any $x = [E]\beta^{-p} \in K^0[\beta,\beta^{-1}]^p_{\mathbb{Q}}(X)$. If $\mathcal{F}^{K^0[\beta,\beta^{-1}]}(x) = \sum_q y_q$, we have from the proof of Theorem 6.2 that

$$y_q = p_{2*}\left(p_1^*([E]\beta^{-p})\frac{(\beta^{-1}\log([\mathfrak{P}^\vee]))^{q+g-p}}{(q+g-p)!}\right) = \frac{\beta^{-q}}{(q+g-p)!}p_{2*}\left([p_1^*E](\log([\mathfrak{P}^\vee]))^{q+g-p}\right),$$

$$\text{and } \hat{\mathcal{F}}^{K^0[\beta,\beta^{-1}]}(y_q) = p_{1*}\left(p_2^*(y_q)\frac{(\beta^{-1}\log([\mathfrak{P}^\vee]))^{p+g-q}}{(p+g-q)!}\right)$$
$$= p_{1*}\left(\frac{\beta^{-q}}{(q+g-p)!}p_2^*p_{2*}([p_1^*E](\log([\mathfrak{P}^\vee]))^{q+g-p})\frac{(\log([\mathfrak{P}^\vee]))^{p+g-q}}{(p+g-q)!}\beta^{q-g-p}\right)$$
$$= \frac{\beta^{-p}}{(q+g-p)!(p+g-q)!}p_{1*}\left(p_2^*p_{2*}([p_1^*E](\log([\mathfrak{P}^\vee]))^{q+g-p})(\log([\mathfrak{P}^\vee]))^{p+g-q}\right).$$

7. Motivic decomposition

Our goal in this section is to get a canonical decomposition of A-motives of abelian varieties as the one constructed for Chow motives by Deninger and Murre in [5]. For an abelian variety X, we want to have a decomposition $h_A(X) = \bigoplus_i h^i_A(X)$ where $h^i_A(X) = (X, p_i, 0)$, p_i being orthogonal projectors such that $\mathbf{n}^*p_i = n^i p_i$. In [16, § 5], Scholl gave an alternative proof of the decomposition for Chow motives and also described the projectors in the decomposition more explicitly.

Let Δ_A denote the class of the diagonal morphism $[X \to X \times X]_A = c_A(\mathrm{id}_X)$ in $A^g_{\mathbb{Q}}(X \times X)$. We are going to show

Theorem 7.1. *There is a canonical decomposition*

$$\Delta_A = \sum_{i=0}^{2g} p_i^A \text{ in } A^g_{\mathbb{Q}}(X\times X)$$

*such that the p_i^A's are mutually orthogonal projectors (*i.e., *$p_i^A \circ p_j^A = 0$ for $i \neq j$) satisfying $(\mathrm{id}_X \times \mathbf{n})^* p_i^A = n^i p_i^A$ for all $n \in \mathbb{Z}$ and for which $c_A(\mathbf{n}) \circ p_i^A = n^i p_i^A = p_i^A \circ c_A(\mathbf{n})$.*

Proof. Note that such a decomposition is unique if it exists. Indeed, if $\{q_i\}_{i=0}^{2g}$ is another such decomposition, then $p_i^A = \sum_{j=0}^{2g} p_i^A \circ q_j$. It suffices to compose with $c_A(\mathbf{n})$ from the left to get $n^i p_i^A = \sum_{j=0}^{2g} n^j p_i^A \circ q_j$, which by substituting the expression for p_i^A gives $\sum_{j=0}^{2g}(n^j - n^i)p_i^A \circ q_j = 0$. Since, this is true for all n, we must have $p_i^A \circ q_j = 0$ for $i \neq j$, implying $p_i^A = p_i^A \circ q_i$. We can similarly show that $q_i = p_i^A \circ q_i$ and therefore the two decompositions coincide.

To see the existence, we consider the following two cases:

Case 1: A is an ordinary theory. As shown in [16, § 5], in $CH^g_{\mathbb{Q}}(X \times X)$, the diagonal Δ_{CH} may be expressed as $\Delta_{\mathrm{CH}} = \sum_{i=0}^{2g} p_i^{\mathrm{can}}$ where for each i, $p_i^{\mathrm{can}} \circ p_i^{\mathrm{can}} = p_i^{\mathrm{can}}$ and $p_i^{\mathrm{can}} \circ p_j^{\mathrm{can}} = 0$ in $\mathrm{Hom}_{Cor^0_{\mathrm{CH}^*_{\mathbb{Q}}}}(X, X)$ for $i \neq j$. Also, $c_{\mathrm{CH}}(\mathbf{n}) \circ p_i^{\mathrm{can}} = n^i p_i^{\mathrm{can}} = p_i^{\mathrm{can}} \circ c_{CH}(\mathbf{n})$.

Now, set $p_i^A := \nu_A^{\mathrm{CH}}(p_i^{\mathrm{can}})$. Note that ν_A^{CH} is a morphism of oriented cohomology theories. Thus, for $\alpha, \beta \in \mathrm{CH}^*_{\mathbb{Q}}(X \times X)$, we have $\nu_A^{\mathrm{CH}}(\alpha \circ \beta) = \nu_A^{\mathrm{CH}}(\alpha) \circ \nu_A^{\mathrm{CH}}(\beta)$, so it readily follows that p_i^A is a projector and $p_i^A \circ p_j^A = 0$ for $i \neq j$. Moreover, since, $\nu_A^{\mathrm{CH}}(c_A(\mathbf{n})) = c_{CH}(\mathbf{n})$, we also get $c_A(\mathbf{n}) \circ p_i^A = n^i p_i^A = p_i^A \circ c_A(\mathbf{n})$ for all $n \in \mathbb{Z}$.

Case 2: A is not ordinary. In this case, since X is an abelian variety, $i^{ad}_{A_{\mathbb{Q}}} : A^*_{ad}(X \times X) \xrightarrow{\sim} A^*_{\mathbb{Q}}(X \times X)$ is a ring isomorphism, that commutes with pushforwards and pullbacks of morphisms between abelian varieties. This implies that for $\alpha, \beta \in A^*_{ad}(X \times X)$, we have $i^{ad}_{A_{\mathbb{Q}}}(\alpha \circ \beta) = i^{ad}_{A_{\mathbb{Q}}}(\alpha) \circ i^{ad}_{A_{\mathbb{Q}}}(\beta)$. Then, taking $p_i^A = i^{ad}_{A_{\mathbb{Q}}}(p_i^{A_{ad}})$ for all i gives us the desired result. □

Let $\mathcal{M}_{A\mathbb{Q}}$ be the category of A-motives over k with $\mathbb{Q}$-coefficients, defined in section 3.2.

Corollary 7.2. *In the category $\mathcal{M}_{A\mathbb{Q}}$, there is a canonical decomposition:*

$$h_A(X) = \bigoplus_{i=0}^{2g} h_A^i(X)\,,$$

where $h_A(X) = (X, \mathrm{id}_X, 0)$ is the motive of X, $h_A^i(X) = (X, p_i^A, 0)$ and the p_i^A's are such that $c_A(\mathbf{n}) \circ p_i^A = n^i p_i^A = p_i^A \circ c_A(\mathbf{n})$.

Proof. This is immediate since, if $c_A(\mathrm{id}_X) = \sum_{i=0}^{n} p_i^A$ for mutually orthogonal projectors p_i^A, then $(X, \mathrm{id}_X, 0) = \bigoplus_{i=0}^n (X, p_i^A, 0)$. □

Let $\nu_A : \Omega^* \to A^*$ denote the canonical morphism. ν_A induces a functor $\widetilde{\nu_A} : \mathcal{M}_\Omega \to \mathcal{M}_A$ of the corresponding categories of motives, acting as ν_A on the morphisms and on objects as $(X, p, m) \mapsto (X, \nu_A(p), m)$.

Corollary 7.3. *We have $\widetilde{\nu_A}(h_\Omega^i(X)) = h_A^i(X)$.*

Proof. It follows from the construction in the proof of Theorem 7.1 that $\nu_A(p_i^\Omega) = p_i^A$ for all i, which implies $\widetilde{\nu_A}(h_\Omega^i(X)) = h_A^i(X)^{\mathrm{can}}$. □

Remark 7.4. We have $(p_i^A)^t = p_{2g-i}^A$ since this property holds for the p_i^{can}s.

Let MU^* be complex cobordism theory. It follows from [8, Theorem B] that in $\mathrm{MU}^*_\mathbb{Q}$, we have the Künneth isomorphism $\mathrm{MU}^*_\mathbb{Q}(Y) \otimes_{\mathbb{L}^*} \mathrm{MU}^*_\mathbb{Q}(Z) \xrightarrow{\alpha} \mathrm{MU}^*_\mathbb{Q}(Y \times Z)$ for smooth projective varieties Y and Z over $\mathbb{C}$.

Proposition 7.5. *Let X be an abelian variety of dimension g over $\mathbb{C}$. If $\alpha^{-1}(\Delta_{\mathrm{MU}}) = \sum_{i+j=2g} a_j \otimes b_i$ with $a_j, b_j \in \mathrm{MU}^j_\mathbb{Q}(X)$, then $p_i^{\mathrm{MU}} = \alpha(a_{2g-i} \otimes b_i)$.*

Proof. For convenience, let us denote $p_i := p_i^{\mathrm{MU}}$ for each $0 \le i \le 2g$. By Theorem 7.1, $\sum_{i=0}^{2g} p_i = \sum_{i=0}^{2g} \alpha(a_{2g-i} \otimes b_i) = \sum_{i=0}^{2g} a_{2g-i} \times b_i$. Applying $(\mathrm{id} \times \mathbf{n})^*$, we get

$$\sum_{i=0}^{2g} n^i p_i = \sum_{i=0}^{2g} a_{2g-i} \times \mathbf{n}^* b_i = \sum_{i=0}^{2g} n^i \left(\sum_{0 \leqslant j \leqslant i} a_{2g-j} \times b_j^i \right)$$

where, by Theorem 6.2, we have $b_i = \sum_{j=i}^{2g} b_i^j$ such that $\mathbf{n}^* b_i^j = n^j b_i^j$. Since the above equality holds for all n, we obtain that $p_i = \sum_{j=0}^{i} a_{2g-j} \times b_j^i$. This implies that $p_{2g-i}^t = \sum_{j=0}^{2g-i} b_j^{2g-i} \times a_{2g-j} = \sum_{j=i}^{2g} b_{2g-j}^{2g-i} \times a_j$. Thus, $p_i = p_{2g-i}^t \Longrightarrow$

$$a_{2g} \times b_0^i + a_{2g-1} \times b_1^i + \cdots + a_{2g-i} \times b_i^i - b_{2g-i}^{2g-i} \times a_i - b_{2g-i-1}^{2g-i} \times a_{i+1} - \cdots - b_0^{2g-i} \times a_{2g} = 0.$$

$$\Longrightarrow a_{2g-j} \times b_j^i = 0, \text{ for } i > j, \text{ and } \quad b_{2g-j}^{2g-i} \times a_j = 0, \text{ for } i < j.$$

Then, $a_{2g-i} \times b_i = \sum_{j=i}^{2g} a_{2g-i} \times b_i^j = a_{2g-i} \times b_i^i = p_i$, which finishes the proof. □

Remark 7.6. The decomposition of the cobordism motive of an abelian variety X is a cobordism-Künneth decomposition, in the sense that $\nu_{\mathrm{MU}}(p_i^\Omega)$ is the corresponding Künneth component of the diagonal in $\mathrm{MU}^*_\mathbb{Q}(X \times X)$.

8. Integral Fourier-Mukai transformation

In this section, we define a Fourier-Mukai transformation on an oriented cohomology theory A^* integrally, without $\mathbb{Q}$-coefficients. We follow the ideas in [2, Proposition 3′]. The key observation is that, for any $x \in A^*(X)$, $\mathcal{F}^A(x)$ may be multiplied with a large enough integer N such that $N\mathcal{F}^A(x) \in A^*(\hat{X})$.

Lemma 8.1. *For every abelian variety X there exists a positive integer N_X, such that for all $x \in A^1(X)$, we have $N_X \exp(\log_A(x)) \in A^*(X)$*

Proof. Let $x \in A^1(X)$. Note that $\forall m > g,\ x^m = 0$. Thus, by Lemma 2.8, $\log_A(x) = \sum_{i=0}^{g-1} \eta_i x^{i+1} \in A^1_{\mathbb{Q}}(X)$. We also get from Lemma 2.8 that $\beta_i = (i+1)\eta_i \in A^{-i}(X)$. Thus,

$$\begin{aligned}\exp(\log_A(x)) =& 1 + \log_A(x) + \frac{\log_A(x)^2}{2!} + \cdots + \frac{\log_A(x)^g}{g!} \\ =& 1 + \sum_{0\leqslant i\leqslant g-1} \frac{\beta_i}{i+1} x^{i+1} + \frac{1}{2!} \sum_{\substack{i,j\geqslant 0\\ i+j\leqslant g-2}} \frac{\beta_i\beta_j}{(i+1)(j+1)} x^{i+j+2} + \\ & + \cdots + \frac{1}{k!} \sum_{\substack{i_j\geqslant 0\\ \sum i_j\leqslant g-k}} \frac{\beta_{i_1}\cdots\beta_{i_k}}{(i_1+1)\cdots(i_k+1)} x^{k+\sum i_j} + \cdots + \frac{x^g}{g!}.\end{aligned}$$

Note that using the Lagrange multiplier rule, we may show that for any $1 \leq k \leq g$, $\left(\frac{g}{k}\right)^k \geq \max_{\substack{i_j\geqslant 0\\ \sum i_j\leqslant g-k}} (i_1+1)\cdots(i_k+1)$. Consider the function $f(x) = \left(\frac{g}{x}\right)^x$ defined on $(0,g]$. Using calculus, it is easy to check that this function attains a maximum at $x = ge^{-1}$. Thus, $e^{g/e} \geq \max_k \left(\frac{g}{k}\right)^k$ implying that

$$\lfloor e^{g/e} \rfloor \geq \max_{1\leqslant k\leqslant g} \max_{\substack{i_j\geqslant 0\\ \sum i_j\leqslant g-k}} (i_1+1)\cdots(i_k+1).$$

Thus, $g!\lfloor e^{g/e}\rfloor!$ is divisible by $k!(i_1+1)\cdots(i_k+1)$ for all k and all sets $\{i_1,\ldots,i_k\}$. This means that we may take $N_X = g!\lfloor e^{g/e}\rfloor!$ so that $N_X \exp(\log_A(x)) \in A^*(X)$. □

Definition 8.2. We define the integral Fourier-Mukai transformation $\mathcal{F}^A_{\mathbb{Z}} : A^*(X) \to A^*(\hat{X})$ as
(i) $\mathcal{F}^A_{\mathbb{Z}} := \mathcal{R}_A(N_{X\times\hat{X}}\mathcal{C}_A(\mathcal{P}))$, *(ii)* $\hat{\mathcal{F}}^A_{\mathbb{Z}} := \mathcal{R}_A(N_{X\times\hat{X}}\mathcal{C}_A(\mathcal{P})^t)$.
Note that, this implies that $\mathcal{F}^A_{\mathbb{Z}}(x) = N_{X\times\hat{X}}\mathcal{F}^A(x)$ and $\hat{\mathcal{F}}^A_{\mathbb{Z}}(y) = N_{X\times\hat{X}}\mathcal{F}^A(y)$.

We get the following easy consequences of the properties of $\mathcal{F}^A$.

Proposition 8.3. *For any $x, y \in A^*(X)$, we have*
(1) $\hat{\mathcal{F}}^A_{\mathbb{Z}} \circ \mathcal{F}^A_{\mathbb{Z}}(x) = N^2_{X\times\hat{X}}(-1)^g\sigma^*_X(x)$, *and* *(2)* $N_{X\times\hat{X}}\mathcal{F}^A_{\mathbb{Z}}(x \star y) = \mathcal{F}^A_{\mathbb{Z}}(x)\mathcal{F}^A_{\mathbb{Z}}(y)$.

Proof. Using the fact that $\mathcal{F}^A_{\mathbb{Z}} = N_{X\times\hat{X}} \cdot \mathcal{F}^A$, (1) follows from Theorem 4.3 and (2) follows from Proposition 5.2, part (1). □

9. Consequences for algebraic cobordism

Let X be an abelian variety over k of dimension g. Let $I \subset CH^g(X)$ denote the set of 0-cycles of degree 0 on X. In [4, § 4], Bloch showed that $I^{\star(r+1)} \star CH^r(X) = (0)$ in the cases $r = 0, 1, g-2, g-1, g$. In [2], Beauville conjectured that

$$(F_p) \qquad \text{For all } x \in CH^p_{\mathbb{Q}}(X),\ \text{we have } \mathcal{F}(x) \in CH^{\geqslant g-p}_{\mathbb{Q}}(\hat{X}).$$

He verified (F_p) for $p = 0, 1, g-2, g-1, g$ ([2, Proposition 8.(i)]) and also showed that

Proposition 9.1 ([2, Proposition 9])**.** *(F_p) implies that $I^{\star(p+1)} \star CH^p(X) = (0)$. In particular, the groups $I^{\star(g+1)}$, $I^{\star g} \star CH^{g-1}(X)$ and $I^{\star(g-1)} \star CH^{g-2}(X)$ are zero.*

We prove an analogue of this Proposition replacing I with numerically trivial cobordism cycles. A notion of numerical equivalence on $\Omega^*(X)$ was defined in [1]. We briefly recall the construction.

Definition 9.2. Let Y be a smooth projective scheme over a field k of characteristic 0. Consider the composition of maps $\Omega^m(Y) \otimes \Omega^n(Y) \to \Omega^{m+n}(Y) \xrightarrow{\pi_{Y*}} \Omega^{m+n-\dim_Y}(k)$, where π_Y is the structure morphism $Y \to \operatorname{Spec} k$. This gives a map of $\mathbb{L}$-modules $\Omega^*(Y) \longrightarrow \operatorname{Hom}_{\mathbb{L}}(\Omega^*(Y), \Omega^*(k))$. We say that a cobordism cycle in $\Omega^*(Y)$ is *numerically equivalent to* 0 if it is in $\mathcal{N}^*(Y)$, which is the kernel of this map. We denote $\Omega^*_{\text{num}}(Y) := \Omega^*(Y)/\mathcal{N}^*(Y)$.

Let $\mathcal{A}^*(Y)$ denote the subgroup of $\Omega^*(Y)$ consisting of cobordism cycles algebraically equivalent to zero. For the definition we refer the reader to [7, § 3]. It follows from [7, Theorem 10.3] and [1, Theorem 5.3] that $\mathcal{A}^*(Y) \subseteq \mathcal{N}^*(Y)$. Let us now go back to the special case of abelian varieties.

Lemma 9.3. *$\mathcal{F}^\Omega_{\mathbb{Z}}$ maps $\mathcal{N}^*(X)$ to $\mathcal{N}^*(\hat{X})$. Hence, $\mathcal{F}^\Omega$ maps $\mathcal{N}^*_{\mathbb{Q}}(X)$ to $\mathcal{N}^*_{\mathbb{Q}}(\hat{X})$.*

Proof. Let $\alpha \in \mathcal{N}^*(X)$ and $N = N_{X\times\hat{X}}$. Then, by definition, $\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha) = p_{\hat{X}*}(p_X^*\alpha \cdot N\mathcal{C}_\Omega(\mathcal{P}))$. Let $\pi_{\hat{X}}$ and π_X be the structure morphisms of $\hat{X}$ and X respectively and let $p_{\hat{X}}$ and p_X be the respective projections of $X \times \hat{X}$ to $\hat{X}$ and X. From the projection formula for any $\gamma \in \Omega^*(\hat{X})$ we obtain

$$\begin{aligned}\pi_{\hat{X}*}(\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha)\cdot\gamma) &= \pi_{\hat{X}*}p_{\hat{X}*}\left(p_X^*\alpha \cdot N\mathcal{C}_\Omega(\mathcal{P}) \cdot p_{\hat{X}}^*\gamma\right)\\ &= \pi_{X*}p_{X*}\left(p_X^*\alpha \cdot N\mathcal{C}_\Omega(\mathcal{P}) \cdot p_{\hat{X}}^*\gamma\right) = \pi_{X*}\left(\alpha\cdot\hat{\mathcal{F}}^\Omega_{\mathbb{Z}}(\gamma)\right).\end{aligned}$$

Thus, numerical triviality of α implies that $\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha)$ is numerically trivial as well. □

Proposition 9.4. *Fix $0 \le p \le g$. If $x \in \Omega^p_{\mathbb{Q}}(X)$ satisfies $\mathcal{F}^\Omega(x) \in \Omega^{\geqslant g-p}_{\mathbb{Q}}(X)$, then*

$$\mathcal{N}^*_{\mathbb{Q}}(X)^{\star(p+1)} \star x = 0.$$

In particular, $\mathcal{N}^*(X)^{\star(g+1)}$ is torsion.

Proof. Pick $\alpha_1, \dots, \alpha_{p+1} \in \mathcal{N}^*(X)$ and let $N = N_{X\times\hat{X}}$. Note that by Proposition 8.3, part (2),

$$N^p\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_1 \star \alpha_2 \star \cdots \star \alpha_{p+1}) = \mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_1)\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_2)\cdots\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_{p+1}). \tag{9.1}$$

By the Generalized degree formula ([10, Theorem 4.4.7]), we get for any i,

$$\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_i) = \deg(\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_i))[\mathrm{Id}_{\hat{X}}] + \sum_{\operatorname{codim}_{\hat{X}} Z>0} \omega_Z[\tilde{Z} \to \hat{X}],$$

where the sum is over closed integral subschemes $Z \subset \hat{X}$, $\tilde{Z}$ is smooth with a birational morphism $\tilde{Z} \to Z$ and $\omega_Z \in \mathbb{L}^*$. Lemma 9.3 shows that $\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_i) \in \mathcal{N}^*_{\mathbb{Q}}(\hat{X})$. Then, by [1, Proposition 3.4], $\deg(\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_i)) = 0$. Hence, $\mathcal{F}^\Omega(\alpha_i) \in \mathbb{L}^* \cdot \Omega^{\geqslant 1}(\hat{X})$.

By (9.1), it follows that $N^p\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_1 \star \cdots \star \alpha_{p+1})$ is in $\mathbb{L}^* \cdot \Omega^{\geqslant p+1}(\hat{X})$. For $p = g$, this implies that $N^g\mathcal{F}^\Omega_{\mathbb{Z}}(\alpha_1 \star \cdots \star \alpha_{g+1}) = 0$. Applying $\hat{\mathcal{F}}^\Omega_{\mathbb{Z}}$, we get by Proposition 8.3, part (1), that $N^{g+2}\alpha_1 \star \cdots \star \alpha_{g+1} = 0$. Thus, $\mathcal{N}^*(X)^{\star(g+1)}$ is torsion.

By Proposition 5.2, part (2), we also get

$$N^p\mathcal{F}^\Omega(\alpha_1 \star \cdots \star \alpha_{p+1} \star x) = N^p\mathcal{F}^\Omega(\alpha_1 \star \cdots \star \alpha_{p+1})\mathcal{F}^\Omega(x) = 0.$$

Applying $\hat{\mathcal{F}}^\Omega$ and using Proposition 5.2, part (1), we get

$$N^p(-1)^g(\sigma_X)^*(\alpha_1 \star \cdots \star \alpha_{p+1} \star x) = 0.$$

Hence, $\alpha_1 \star \cdots \star \alpha_{p+1} \star x = 0 \in \Omega^*_{\mathbb{Q}}(X)$, which completes the proof. □

Corollary 9.5. $\mathcal{A}^*(X)^{\star(g+1)} = 0$.

Proof. Since $\mathcal{A}^*(X) \subseteq \mathcal{N}^*(X)$, we get by Proposition 9.4 that $N^{g+2}_{X\times\hat{X}}\mathcal{A}^*(X)^{\star(g+1)} = 0$. But, [7, Theorem 8.4] implies that $\mathcal{A}^*(X)$ is divisible. Thus, $\mathcal{A}^*(X)^{\star(g+1)} = 0$. □

Remark 9.6. One can check that $\mathcal{N}^*(X)$ forms an ideal of $\Omega^*(X)$ under Pontryagin product. By [10, Lemma 4.5.10] and [1, Theorem 3.2], $\mathcal{N}^g(X) \xrightarrow{\sim} I$, which is the subgroup of 0-cycles of degree 0. This implies $\mathcal{N}^g(X)/\mathcal{N}^g(X)^{\star 2} \xrightarrow{\sim} I/I^{\star 2} \xrightarrow{\sim} X$. It would be interesting to study the structure of the groups $\mathcal{N}^p(X)/\mathcal{N}^p(X)^{\star 2}$ for $p < g$.

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